

# CONFORMALLY FEDOSOV MANIFOLDS

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ABSTRACT. We introduce the notion of a conformally Fedosov structure and construct an associated Cartan connection.

## 1. INTRODUCTION

On a Riemannian manifold there is a unique torsion-free connection on the tangent bundle preserving the metric. Known as the *Levi Civita* connection, it is the basis for doing calculus and understanding the geometry on such a manifold. On a symplectic manifold, there is no preferred connection. Instead, there are many *symplectic connections*, torsion-free connections preserving the symplectic form. Choosing one defines what is known as a *Fedosov manifold* [8].

On a conformal manifold there is no preferred connection on the tangent bundle: each metric in the conformal class gives rise to its own Levi-Civita connection. Instead, a conformal structure induces a canonically defined *Cartan connection* [5, §1.6.7]. It is the basic object in conformal differential geometry, can be regarded as a connection on an auxiliary vector bundle [3], and whose curvature provides the first conformal invariant.

On a *projective* manifold [5, §4.1.5], there is an equivalence class of torsion-free connections on the tangent bundle. Again, it is the Cartan connection, built from these affine connections, which may equivalently be regarded [3] as a connection on some auxiliary vector bundle and whose curvature is the basic projective invariant.

On a *conformally symplectic* manifold [13] there is a local equivalence class of symplectic forms defined only up to scale. In this article we shall show that one can combine projective differential geometry with the notion of a Fedosov manifold to obtain what we shall call *conformally Fedosov* manifolds. They are obtained by adding further structure to a conformally symplectic manifold and have the remarkable property that a canonical Cartan connection can then be constructed. This lies outside the realm of parabolic differential geometry [5].

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**1.1. Notation and terminology.** Notice that we are choosing to write ‘conformally Fedosov’ rather than ‘locally conformally Fedosov’ or ‘locally conformal Fedosov.’ These various alternatives are regularly employed in the context of Kähler or symplectic geometry. Our usage is chosen for several reasons. Firstly, we suppress the word ‘local’ in our terminology. In this we follow by analogy the standard convention that the round metric on the sphere is ‘conformally flat’ rather than being ‘locally conformally flat,’ for example. Secondly, our terminology is reasonably succinct. Thirdly, for Kähler geometry the corresponding terminology of ‘conformally Kähler’ was introduced by Westlake [14] already in 1954.

We shall often need to manipulate tensors on a smooth manifold and, for this purpose, we shall use Penrose’s abstract index notation [11]. In brief, covariant tensors will be decorated with subscripts, contravariant tensors with superscripts, and the natural pairing between vectors and 1-forms by repeating an index  $X^a\omega_a$  in accordance with the ‘Einstein summation convention.’ For any tensor  $\phi_{abc}$  we shall write  $\phi_{(abc)}$  for its symmetric part and  $\phi_{[abc]}$  for its skew part. For example, to say that  $\omega_{ab}$  is a 2-form is to say that  $\omega_{ab} = \omega_{[ab]}$  or, equivalently, that  $\phi_{(ab)} = 0$  and, for any torsion-free connection  $\nabla_a$ , the expressions

$$\nabla_{[a}\omega_{bc]} \quad \text{and} \quad X^a\nabla_a\omega_{bc} - 2(\nabla_{[b}X^a)\omega_{c]a}$$

deliver the exterior derivative of  $\omega_{ab}$  and the Lie derivative of  $\omega_{ab}$  in the direction of the vector field  $X^a$ , respectively.

We shall write  $\Lambda^p$  for the bundle of  $p$ -forms on a smooth manifold, suppressing the name of the manifold itself. The exterior derivative will be denoted by  $d : \Lambda^p \rightarrow \Lambda^{p+1}$ .

## 2. CONFORMALLY SYMPLECTIC MANIFOLDS

In the first instance, a *conformally symplectic* manifold [1, 13] is an even-dimensional manifold  $M$  of dimension at least four equipped with a non-degenerate 2-form  $J$  such that

$$(1) \quad dJ = 2\alpha \wedge J$$

for some closed 1-form  $\alpha$ . Non-degeneracy of  $J$  ensures  $J : \Lambda^1 \rightarrow \Lambda^3$  is injective whence  $\alpha$  is uniquely determined by  $J$ , should such a 1-form exist. It is called the *Lee form* [9] and, in case  $\dim M \geq 6$  we see that

$$0 = d^2J = 2d\alpha \wedge J + 2\alpha \wedge dJ = 2d\alpha \wedge J + 4\alpha \wedge \alpha \wedge J = 2d\alpha \wedge J$$

and, as  $J \wedge \_ : \Lambda^2 \rightarrow \Lambda^4$  is injective, closure of  $\alpha$  is automatic. If we rescale  $J$  by a positive smooth function, say  $\hat{J} = \Omega^2 J$ , then (1) remains valid with  $\alpha$  replaced by  $\hat{\alpha} = \alpha + \Upsilon$  for  $\Upsilon \equiv d \log \Omega$ . Hence,

the notion of conformally symplectic is invariant under such rescalings (and also in dimension 4 since  $d\Upsilon = 0$ ). Locally, we may use this freedom to eliminate  $\alpha$  and obtain an ordinary symplectic structure. Globally, however, this need not be the case. For example, the rescaled symplectic form

$$J \equiv (1/\|x\|)^2 (dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \dots)$$

on  $\mathbb{R}^{2n}$  is invariant under dilation  $x \mapsto \lambda x$  and, therefore, descends to a conformally symplectic structure on  $S^1 \times S^{2n-1}$  whereas there is no global symplectic form on this manifold.

More precisely, a *conformally symplectic* manifold is a pair  $(M, [J])$  where  $[J]$  is an equivalence class of non-degenerate 2-forms satisfying (1) where  $J$  and  $\hat{J}$  are said to be equivalent if and only if  $\hat{J} = \Omega^2 J$  for some positive smooth function  $\Omega$ . As one often does in conformal geometry in which a Riemannian metric is only defined up to local rescaling  $g \mapsto \hat{g} = \Omega^2 g$ , it is usual to pick a representative  $J$  and work with that representative, whilst checking that one's conclusions are independent of this choice. The basic example of this approach is in noting that the requirement (1) is itself independent of such a choice.

### 3. PROJECTIVE MANIFOLDS

A *projective structure* [6] on a manifold  $M$  is an equivalence class of torsion-free affine connections on  $M$ , where two connections  $\nabla_a$  and  $\hat{\nabla}_a$  are said to be projectively equivalent if and only if

$$(2) \quad \hat{\nabla}_a \phi_b = \nabla_a \phi_b - \nu_a \phi_b - \nu_b \phi_a$$

for some 1-form  $\nu_a$ .

**Lemma 1.** *If  $J_{ab}$  is skew, then*

$$\hat{\nabla}_{(a} J_{b)c} = \nabla_{(a} J_{b)c} - 3\nu_{(a} J_{b)c}.$$

*Proof.* The Leibniz rule extends (2) to all other tensors. Thus,

$$\hat{\nabla}_a J_{bc} = \nabla_a J_{bc} - 2\nu_a J_{bc} - \nu_b J_{ac} - \nu_c J_{ba}$$

and symmetrising over  $ab$  gives the desired conclusion.  $\square$

**Proposition 1.** *If  $J_{ab}$  is skew, then the requirement that*

$$(3) \quad \nabla_{(a} J_{b)c} = \beta_{(a} J_{b)c}$$

*for some 1-form  $\beta_a$  is projectively invariant.*

*Proof.* From Lemma 1, if  $\nabla_a$  is replaced by  $\hat{\nabla}_a$  according to (2), then (3) still holds but with  $\beta_a$  replaced by  $\hat{\beta}_a \equiv \beta_a - 3\nu_a$ .  $\square$

## 4. CONFORMALLY FEDOSOV MANIFOLDS

Let  $(M, [J])$  be a conformally symplectic manifold. We may express the requirement (1) in terms of any torsion-free connection  $\nabla_a$  as

$$(4) \quad \nabla_{[a} J_{bc]} = 2\alpha_{[a} J_{bc]}.$$

Now let us also insist on (3). As observed in Proposition 1, this is only a restriction on the projective class of  $\nabla_a$ . We may assemble these conditions into the following formal definition. A *conformally Fedosov* manifold is a triple  $(M, [J], [\nabla])$  where

- $M$  is a smooth manifold of dimension  $2n \geq 4$ ,
- $[J]$  is an equivalence class of non-degenerate 2-forms defined up to rescaling  $J \mapsto \hat{J} = \Omega^2 J$  for some positive function  $\Omega$ ,
- $[\nabla]$  is a projective structure, i.e. an equivalence class of torsion-free connections defined up to (2) for some 1-form  $\nu_a$ ,
- the following equations hold

$$(5) \quad \nabla_{[a} J_{bc]} = 2\alpha_{[a} J_{bc]} \quad \nabla_{[a} \alpha_{b]} = 0 \quad \nabla_{(a} J_{b)c} = \beta_{(a} J_{b)c},$$

for some 1-forms  $\alpha_a$  and  $\beta_a$ .

We have already observed that the requirement (4) depends only on the conformal class of  $J_{ab}$  (and that  $\nabla_{[a} \alpha_{b]} = 0$  is automatic for  $n \geq 3$ ). Proposition 1 says that (3) is projectively invariant. Finally, to make sure that this definition makes sense, let us observe that (3) is also conformally invariant: if  $\hat{J}_{ab} = \Omega^2 J_{ab}$ , then (3) continues to hold but with  $\beta_a$  replaced by  $\hat{\beta}_a = \beta_a + 2\Upsilon_a$ .

We shall often have occasion to ‘raise and lower’ indices using  $J_{ab}$  and its inverse  $J^{ab}$ . Specifically, let  $J_{ac} J^{bc} = \delta_a^b$ , where  $\delta_a^b$  is the Kronecker delta. We then decree that

$$\phi^a \equiv J^{ab} \phi_b \quad \psi_b \equiv \psi^a J_{ab}$$

and henceforth freely make use of these options without comment.

**Proposition 2.** *Let  $(M, [J], [\nabla])$  be a conformally Fedosov manifold. Any representatives  $J_{ab}$  and  $\nabla_a$  of the structure uniquely determine the 1-forms  $\alpha_a$  and  $\beta_a$  occurring in (5) and, conversely,*

$$(6) \quad \nabla_a J_{bc} = 2\alpha_{[a} J_{bc]} + \frac{2}{3}\beta_{(a} J_{b)c} - \frac{2}{3}\beta_{(a} J_{c)b}$$

*determines the full covariant derivative  $\nabla_a J_{bc}$ .*

*Proof.* Let  $J^{ab}$  denote the inverse of  $J_{ab}$ . Then the identities

$$3J^{bc}\nabla_{[a} J_{bc]} = 2(n-2)\alpha_a \quad \text{and} \quad 2J^{bc}\nabla_{(a} J_{b)c} = (2n+1)\beta_a$$

readily follow from (5). Conversely, expanding the right hand side of

$$2\alpha_{[a} J_{bc]} + \frac{2}{3}\beta_{(a} J_{b)c} - \frac{2}{3}\beta_{(a} J_{c)b} = \nabla_{[a} J_{bc]} + \frac{2}{3}\nabla_{(a} J_{b)c} - \frac{2}{3}\nabla_{(a} J_{c)b}$$

gives  $\nabla_a J_{bc}$ , as required.  $\square$

**Proposition 3.** *For any conformally Fedosov manifold  $(M, [J], [\nabla])$ , if a representative 2-form  $J_{ab}$  is chosen, then there is a unique torsion-free connection in the projective class such that*

$$(7) \quad \nabla_a J_{bc} = 2J_{a[b}\alpha_{c]}.$$

*Proof.* When the connection  $\nabla_a$  is replaced by  $\hat{\nabla}_a$  according to (2), the 1-form  $\alpha_a$  does not change but  $\beta_a$  is replaced by  $\hat{\beta}_a = \beta_a - 3\nu_a$ . Therefore, we can uniquely arrange that  $\alpha_a + \beta_a = 0$ , in which case (6) implies that

$$\nabla_a J_{bc} = 2\alpha_{[a}J_{bc]} - \frac{2}{3}\alpha_{(a}J_{b)c} + \frac{2}{3}\alpha_{(a}J_{c)b}$$

and expanding the right hand side gives  $2J_{a[b}\alpha_{c]}$ , as required.  $\square$

In view of this Proposition, an alternative definition of a conformally Fedosov manifold is as follows. Firstly, define an equivalence relation on pairs  $(J, \nabla)$  consisting of a non-degenerate symplectic form  $J_{ab}$  and a torsion-free connection  $\nabla_a$  by allowing simultaneous replacements

$$(8) \quad \begin{aligned} J_{ab} &\mapsto \hat{J}_{ab} = \Omega^2 J_{ab} \\ \nabla_a \phi_b &\mapsto \hat{\nabla}_a \phi_b = \nabla_a \phi_b - \Upsilon_a \phi_b - \Upsilon_b \phi_a, \end{aligned} \quad \text{where } \Upsilon_a = \nabla_a \log \Omega.$$

Writing  $[J, \nabla]$  for the equivalence class of such pairs, a conformally Fedosov manifold may then be defined as a pair  $(M, [J, \nabla])$  such that (7) holds (and one can check directly that (7) is invariant under (8) if one decrees that  $\alpha_a \mapsto \hat{\alpha}_a = \alpha_a + \Upsilon_a$ ). For the rest of this article we shall adopt this alternative definition of a conformally Fedosov manifold. By analogy with ordinary conformal structures, we shall refer to the pair  $[J, \nabla]$  as a *conformal class*.

**Proposition 4.** *Any conformally symplectic manifold  $(M, [J])$  can be extended to a conformally Fedosov structure  $(M, [J, \nabla])$ .*

*Proof.* Pick a representative 2-form  $J_{ab}$ . We are required to find a torsion-free connection  $\nabla_a$  such that (7) is satisfied for some 1-form  $\alpha_a$ . Recall that the 1-form  $\alpha_a$  is already determined by (4) independent of choice of  $\nabla_a$ . Locally, there is no problem in finding a suitable  $\nabla_a$ : choose  $\Omega$  such that  $\hat{J}_{ab} = \Omega^2 J_{ab}$  is closed and define  $\nabla_a$  by (8) where  $\hat{\nabla}_a$  is the flat connection in Darboux coördinates for  $\hat{J}_{ab}$ . We may use a partition of unity to patch these connections together.  $\square$

**Proposition 5.** *Equation (7) is equivalent to*

$$(9) \quad \nabla_a J^{bc} = 2\alpha^{[b}\delta_a^{c]},$$

where  $\alpha^b \equiv J^{bc}\alpha_c$ .

*Proof.* Recall that  $J_{bc}J^{bd} = \delta_c^d$ . Differentiating this and substituting from (7) gives

$$0 = J_{bc}\nabla_a J^{bd} + 2J_{a[b}\alpha_{c]}J^{bd} = J_{bc}\nabla_a J^{bd} - \delta_a^d\alpha_c + J_{ac}\alpha^d.$$

Therefore,

$$\nabla_a J^{bc} = J^{be}J_{de}\nabla_a J^{dc} = J^{be}(\delta_a^c\alpha_e - J_{ae}\alpha^c) = \delta_a^c\alpha^b - \delta_a^b\alpha^c,$$

as required.  $\square$

**Corollary 1.** *A projective structure  $[\nabla]$  cannot necessarily be extended to a conformally Fedosov structure.*

*Proof.* That equation (9) hold for some vector field  $\alpha^a$  is equivalent to requiring that

$$\text{the trace-free part of } (\nabla_a J^{bc}) = 0,$$

which is a system of finite type as explained in [2]. Hence, there are obstructions to its solution (and writing it as (9) is the first step in its prolongation).  $\square$

## 5. CURVATURE

For any torsion-free affine connection  $\nabla_a$ , the curvature  $R_{ab}{}^c{}_d$  of  $\nabla_a$  is characterised by the equation

$$(10) \quad (\nabla_a\nabla_b - \nabla_b\nabla_a)X^c = R_{ab}{}^c{}_dX^d.$$

Recall that we are free to ‘lower an index’ and write the curvature as  $R_{abcd}$  in the presence of a non-degenerate 2-form  $J_{ab}$ .

**Theorem 1.** *Choosing any representative 2-form  $J_{ab}$  and connection  $\nabla_a$  of a conformally Fedosov manifold  $(M, [J, \nabla])$ , the curvature  $R_{ab}{}^c{}_d$  of  $\nabla_a$  may be uniquely written as*

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + \delta_a^c P_{bd} - \delta_b^c P_{ad},$$

where  $P_{ab}$  is a symmetric tensor and  $W_{ab}{}^c{}_d$  satisfies

$$(11) \quad W_{ab}{}^c{}_d = W_{[ab]}{}^c{}_d \quad W_{[ab]}{}^c{}_d = 0 \quad W_{ab}{}^a{}_d = 0.$$

Under conformal rescaling (8), the tensor  $W_{ab}{}^c{}_d$  is unchanged whilst

$$\hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b.$$

Furthermore, the tensor  $W_{abcd}$  may be uniquely decomposed as

$$W_{abcd} = V_{abcd} - \frac{3}{2n-1}J_{ac}\Phi_{bd} + \frac{3}{2n-1}J_{bc}\Phi_{ad} + J_{ad}\Phi_{bc} - J_{bd}\Phi_{ac} + 2J_{ab}\Phi_{cd},$$

where

$$(12) \quad V_{abcd} = V_{[ab](cd)} \quad V_{[abc]d} = 0 \quad J^{ab}V_{abcd} = 0$$

and  $\Phi_{ab}$  is symmetric.

*Proof.* The curvature of any torsion-free connection may be uniquely and conveniently written as

$$(13) \quad R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + 2\delta_{[a}{}^c P_{b]d} + \beta_{ab}\delta^c{}_d$$

where  $W_{ab}{}^c{}_d$  satisfies (11) and  $\beta_{ab} = -2P_{[ab]}$ . Let us suppose, for the moment, that  $\nabla_a J_{bc} = 0$ . Then, together with the Bianchi identity, we have

$$(14) \quad R_{abcd} = R_{[ab](cd)} \quad \text{and} \quad R_{[abc]d} = 0,$$

corresponding to an irreducible representation of  $\text{SL}(2n, \mathbb{R})$ . Branching this representation under  $\text{Sp}(2n, \mathbb{R}) \subset \text{SL}(2n, \mathbb{R})$  gives

$$(15) \quad R_{abcd} = V_{abcd} + J_{ac}\Phi_{bd} - J_{bc}\Phi_{ad} + J_{ad}\Phi_{bc} - J_{bd}\Phi_{ac} + 2J_{ab}\Phi_{cd},$$

where  $V_{abcd}$  satisfies (12) and  $\Phi_{ab}$  is symmetric. From (13) we see that

$$J^{cd}R_{abcd} = J^{cd}(W_{abcd} + J_{ac}P_{bd} - J_{bc}P_{ad} - \beta_{ab}J_{cd}) = -(2n+1)\beta_{ab}$$

whereas (14) implies that  $J^{cd}R_{abcd}$  should vanish. Therefore  $\beta_{ab} = 0$  and consequently  $P_{ab}$  is symmetric. Thus, we have

$$(16) \quad \begin{aligned} R_{abcd} &= W_{abcd} + J_{ac}P_{bd} - J_{bc}P_{ad} \\ &= V_{abcd} + J_{ac}\Phi_{bd} - J_{bc}\Phi_{ad} + J_{ad}\Phi_{bc} - J_{bd}\Phi_{ac} + 2J_{ab}\Phi_{cd} \end{aligned}$$

from (13) and (15), respectively. Now computing  $J^{bc}R_{abcd}$  from each of these two decompositions gives  $(2n-1)P_{ad} = 2(n+1)\Phi_{ad}$ . Substituting back and rearranging the result gives the decomposition of  $W_{abcd}$  as in the statement of the theorem.

This was all under the assumption that  $\nabla_a J_{bc} = 0$  and locally, there is always a connection  $\nabla_a$  and 2-form  $J_{ab}$  in the conformal class  $[J, \nabla]$  for which this assumption is valid. In general, we must see how our conclusions are affected by a conformal change (8). The decomposition (13) is familiar from projective differential geometry [6] and, since (8) is controlled by a *closed* 1-form  $\Upsilon_a$ , we have  $\hat{\beta}_{ab} = 0$  whilst

$$\hat{W}_{ab}{}^c{}_d = W_{ab}{}^c{}_d \quad \text{and} \quad \hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b.$$

Finally, having a lowered index, we see that  $\hat{W}_{abcd} = \Omega^2 W_{abcd}$  and the algebraic decomposition of  $W_{abcd}$  given in the statement of the theorem remains valid with

$$\hat{V}_{abcd} = \Omega^2 V_{abcd} \quad \text{and} \quad \hat{\Phi}_{ab} = \Phi_{ab}.$$

This completes the proof.  $\square$

On a conformally Fedosov manifold, although  $J_{ab}$  is only defined up to scale, the local stipulation that  $\nabla_a J_{bc} = 0$  for some torsion-free connection  $\nabla_a$  in the projective class characterises a globally defined affine connection whose curvature decomposes as (15) (also depending

only on  $J_{ab}$  up to scale). More generally, the proof of Theorem 1 decomposes the curvature into three  $\mathrm{Sp}(2n, \mathbb{R})$ -irreducible parts,

$$V_{abcd} \in \overset{2}{\bullet} \overset{1}{\bullet} \overset{0}{\bullet} \dots \overset{0}{\bullet} \overset{0}{\bullet} \quad \Phi_{ab} \in \overset{2}{\bullet} \overset{0}{\bullet} \overset{0}{\bullet} \dots \overset{0}{\bullet} \overset{0}{\bullet} \quad P_{ab} \in \overset{2}{\bullet} \overset{0}{\bullet} \overset{0}{\bullet} \dots \overset{0}{\bullet} \overset{0}{\bullet}$$

according to

$$(17) \quad R_{abcd} = V_{abcd} + 2J_{ab}\Phi_{cd} - 2\Phi_{c[a}J_{b]d} + \frac{6}{2n-1}J_{c[a}\Phi_{b]d} - 2J_{c[a}P_{b]d}$$

and under conformal change (8), we have

$$\hat{V}_{abcd} = \Omega^2 V_{abcd} \quad \hat{\Phi}_{ab} = \Phi_{ab} \quad \hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b.$$

It is easy to give explicit formulæ for these parts, viz.:-

$$P_{bd} = \frac{1}{2n-1} J^{ac} R_{abcd} \quad \Phi_{cd} = \frac{2n-1}{8(n+1)(n-1)} (J^{ab} R_{abcd} - 2P_{cd})$$

and  $V_{abcd}$  is then determined by (17). As an example, the curvature of  $\mathbb{CP}_n$  with its standard Fubini-Study metric is given by

$$R_{abcd} = g_{bd}J_{ac} - g_{ad}J_{bc} - g_{ac}J_{bd} + g_{bc}J_{ad} + 2J_{ab}g_{cd}$$

and one easily computes that

$$P_{ab} = \frac{2(n+1)}{2n-1} g_{ab} \quad \Phi_{ab} = g_{ab} \quad V_{abcd} = 0.$$

As in the proof of Theorem 1, it is often convenient locally to work in a gauge in which  $\alpha_a = 0$  for then  $\nabla_a J_{bc} = 0$  and the curvature  $R_{abcd}$  decomposes according to (15). Also recall from (16) that

$$(18) \quad (2n-1)P_{ab} = 2(n+1)\Phi_{ab}.$$

We shall refer to a choice of pair  $(J_{ab}, \nabla_a)$  from a conformally Fedosov structure  $[J_{ab}, \nabla_a]$  for which  $\nabla_a J_{bc} = 0$  as a *Fedosov gauge*. This is in accordance with the notion of Fedosov manifold [8]. We pause here to examine some consequences of the Bianchi identity  $\nabla_{[e} R_{ab]cd} = 0$ . From (15) we conclude that

$$\begin{aligned} 0 &= 3J^{de}\nabla_{[e} R_{ab]cd} \\ &= \nabla^d V_{abcd} + J_{ac}\nabla^d \Phi_{bd} - J_{bc}\nabla^d \Phi_{ad} + \nabla_a \Phi_{bc} - \nabla_b \Phi_{ac} \\ &\quad - \nabla_a \Phi_{bc} - 2n\nabla_b \Phi_{ac} - \nabla_a \Phi_{bc} + \nabla_b \Phi_{ac} + 2n\nabla_a \Phi_{bc} + \nabla_b \Phi_{ac} \\ &\quad + 2J_{ab}\nabla^d \Phi_{cd} + 2\nabla_a \Phi_{bc} - 2\nabla_b \Phi_{ac} \\ &= \nabla^d V_{abcd} + J_{ac}\nabla^d \Phi_{bd} - J_{bc}\nabla^d \Phi_{ad} + 2J_{ab}\nabla^d \Phi_{cd} \\ &\quad + (2n+1)\nabla_a \Phi_{bc} - (2n+1)\nabla_b \Phi_{ac}. \end{aligned}$$

This suggests that one introduce the tensor

$$Y_{abc} \equiv \nabla_a \Phi_{bc} - \nabla_b \Phi_{ac} + \frac{1}{2n+1} (J_{ac}\nabla^d \Phi_{bd} - J_{bc}\nabla^d \Phi_{ad} + 2J_{ab}\nabla^d \Phi_{cd}),$$

noting that

$$Y_{abc} = Y_{[ab]c} \quad Y_{[abc]} = 0 \quad J^{ab}Y_{abc} = 0.$$



We have established the contracted Bianchi identity

$$(19) \quad \nabla^d V_{abcd} + (2n+1)Y_{abc} = 0.$$

For later use, it is convenient to introduce the tensor  $S_a \equiv \frac{1}{2n+1} \nabla^b \Phi_{ab}$  so that

$$(20) \quad Y_{abc} = \nabla_a \Phi_{bc} - \nabla_b \Phi_{ac} + J_{ac} S_b - J_{bc} S_a + 2J_{ab} S_c.$$

## 6. THE TRACTOR CONNECTION IN CONFORMAL GEOMETRY

Here we review the construction of the conformal tractor bundle and its connection following the conventions of [3, 6]. We omit all details. The purpose of this section is to establish notation and to motivate the corresponding construction in the conformally Fedosov setting.

Firstly, we recall that the bundle  $\Lambda^0[w]$  of *conformal densities of weight  $w$*  is defined as the trivial bundle  $\Lambda^0$  in the presence of a chosen metric  $g_{ab}$  in the conformal class  $[g_{ab}]$ . Its smooth sections may then be identified as smooth functions but if a different metric is chosen, say  $\hat{g}_{ab} = \Omega^2 g_{ab}$ , then the corresponding functions are obliged to change according to  $\hat{\sigma} = \Omega^w \sigma$ . A similar notion applies to tensors and tensor bundles. In particular, a 1-form of weight  $w$  transforms according to  $\hat{\omega}_a = \Omega^w \omega_a$  when  $g_{ab}$  is replaced by  $\hat{g}_{ab} = \Omega^2 g_{ab}$  and the corresponding bundle is denoted  $\Lambda^1[w]$ . Tautologically, the conformal metric itself is a symmetric covariant 2-form of conformal weight 2. The *standard tractor bundle*  $\mathbb{T}$  is defined in the presence of a chosen metric  $g_{ab}$  to be the direct sum

$$\mathbb{T} = \Lambda^0[1] \oplus \Lambda^1[1] \oplus \Lambda^0[-1]$$

but if the metric is rescaled as  $\hat{g}_{ab} = \Omega^2 g_{ab}$ , then this decomposition is mandated to change according to

$$\begin{bmatrix} \hat{\sigma} \\ \hat{\mu}_b \\ \hat{\rho} \end{bmatrix} = \begin{bmatrix} \sigma \\ \mu_b + \Upsilon_b \sigma \\ \rho - \Upsilon^b \mu_b - \frac{1}{2} \Upsilon^b \Upsilon_b \sigma \end{bmatrix}, \quad \text{where } \Upsilon_a \equiv \nabla_a \log \Omega.$$

For a chosen metric  $g_{ab}$  in the conformal class, the *tractor connection* is defined by

$$\nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab} \rho + P_{ab} \sigma \\ \nabla_a \rho - P_a^b \mu_b \end{bmatrix},$$

where  $\nabla_a \mu_b$  is the Levi-Civita connection of  $g_{ab}$ . One checks that this definition is conformally invariant. As detailed in [3], this construction is essentially due to T.Y. Thomas [12] and is equivalent to the Cartan connection [4] constructed three years earlier.

## 7. A CONFORMALLY FEDOSOV TRACTOR CONNECTION

Firstly, we shall build a *tractor bundle* on a conformally Fedosov manifold, a vector bundle which we shall then endow with a canonically defined connection. As usual, given a conformally Fedosov manifold  $(M, [J, \nabla])$ , definitions will be given in terms of chosen representatives  $J_{ab}$  and  $\nabla_a$  and then we shall check that these definitions respect the allowed freedom (8). Firstly, we define the bundle  $\Lambda^0[w]$  of *conformal densities of weight  $w$*  as the trivial bundle in the presence of chosen representatives  $(J_{ab}, \nabla_a)$  but, under the allowed replacements (8), its sections regarded as functions are decreed to change by  $\hat{\sigma} = \Omega^w \sigma$ .

For chosen representatives, the vector bundle  $\mathbb{T}$  is defined as

$$\mathbb{T} = \Lambda^0[1] \oplus \Lambda^1[1] \oplus \Lambda^0[-1]$$

but this splitting is decreed to change as

$$(21) \quad \begin{bmatrix} \hat{\sigma} \\ \hat{\mu}_b \\ \hat{\rho} \end{bmatrix} = \begin{bmatrix} \sigma \\ \mu_b + \Upsilon_b \sigma \\ \rho - \Upsilon^b \mu_b + \Upsilon^b \alpha_b \sigma \end{bmatrix}$$

under (8), where  $\alpha_a$  is defined by (7). We may check this decree is self-consistent as follows.

$$\begin{aligned} \hat{\hat{\sigma}} &= \hat{\sigma} = \sigma, \\ \hat{\hat{\mu}}_b &= \hat{\mu}_b + \hat{\Upsilon}_b \hat{\sigma} = \mu_b + \Upsilon_b \sigma + \hat{\Upsilon}_b \sigma = \mu_b + (\Upsilon_b + \hat{\Upsilon}_b) \sigma, \\ \hat{\hat{\rho}} &= \hat{\rho} - \hat{\Upsilon}^b \hat{\mu}_b + \hat{\Upsilon}^b \hat{\alpha}_b \hat{\sigma} \\ &= \rho - \Upsilon^b \mu_b + \Upsilon^b \alpha_b \sigma - \hat{\Upsilon}^b (\mu_b + \Upsilon_b \sigma) + \hat{\Upsilon}^b (\alpha_b + \Upsilon_b) \sigma \\ &= \rho - (\Upsilon^b + \hat{\Upsilon}^b) \mu_b + (\Upsilon^b + \hat{\Upsilon}^b) \alpha_b \sigma. \end{aligned}$$

There is a non-degenerate skew form defined on  $\mathbb{T}$  by

$$(22) \quad \left\langle \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix}, \begin{bmatrix} \tilde{\sigma} \\ \tilde{\mu}_c \\ \tilde{\rho} \end{bmatrix} \right\rangle = \sigma \tilde{\rho} - J^{bc} \mu_b \tilde{\mu}_c - \rho \tilde{\sigma} = \sigma \tilde{\rho} + \mu^b \tilde{\mu}_b - \rho \tilde{\sigma},$$

(which one readily checks is preserved by (21)).

Although not yet the tractor connection, consider the connection  $D_a$  on  $\mathbb{T}$  defined by

$$(23) \quad D_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b - J_{ab} \rho + P_{ab} \sigma - J_{ab} \alpha^c \mu_c \\ \nabla_a \rho - P_a^b \mu_b - (2\alpha^b P_{ab} + \alpha^b \nabla_a \alpha_b) \sigma \end{bmatrix}.$$

**Proposition 6.** *The connection (23) is well-defined, i.e. is independent of choice of representatives  $(J_{ab}, \nabla_a)$ , and preserves the skew form (22).*

*Proof.* Recall that (7) can be rewritten according to Proposition 5 as

$$\nabla_a J^{bc} = 2\alpha^{[b}\delta_a^{c]}.$$

We shall show in Lemma 2 below that this leads to

$$(24) \quad \nabla_a \Upsilon^b = J^{bc} \nabla_a \Upsilon_c + \Upsilon_a \alpha^b + \Upsilon^c \alpha_c \delta_a^b.$$

For convenience, let

$$T_a \equiv 2\alpha^b P_{ab} + \alpha^b \nabla_a \alpha_b.$$

Now we compute

$$\begin{aligned} \hat{D}_a \begin{bmatrix} \hat{\sigma} \\ \hat{\mu}_b \\ \hat{\rho} \end{bmatrix} &= \begin{bmatrix} \hat{\nabla}_a \hat{\sigma} - \hat{\mu}_a \\ \hat{\nabla}_a \hat{\mu}_b - \hat{J}_{ab} \hat{\rho} + \hat{P}_{ab} \hat{\sigma} - \hat{J}_{ab} \hat{\alpha}^c \hat{\mu}_c \\ \hat{\nabla}_a \hat{\rho} + \hat{P}_{ab} \hat{\mu}^b - \hat{T}_a \hat{\sigma} \end{bmatrix} \\ &= \begin{bmatrix} \hat{\nabla}_a \sigma - (\mu_a + \Upsilon_a \sigma) \\ \hat{\nabla}_a (\mu_b + \Upsilon_b \sigma) - J_{ab} (\rho - \Upsilon^c \mu_c + \Upsilon^c \alpha_c \sigma) \\ \quad + (P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b) \sigma - J_{ab} (\alpha^c + \Upsilon^c) (\mu_c + \Upsilon_c \sigma) \\ \hat{\nabla}_a (\rho - \Upsilon^c \mu_c + \Upsilon^c \alpha_c \sigma) \\ \quad + (P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b) (\mu^b + \Upsilon^b \sigma) - \hat{T}_a \sigma \end{bmatrix} \\ &= \begin{bmatrix} \nabla_a \sigma + \Upsilon_a \sigma - (\mu_a + \Upsilon_a \sigma) \\ \nabla_a (\mu_b + \Upsilon_b \sigma) - \Upsilon_b (\mu_a + \Upsilon_a \sigma) - J_{ab} (\rho - \Upsilon^c \mu_c + \Upsilon^c \alpha_c \sigma) \\ \quad + (P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b) \sigma - J_{ab} (\alpha^c + \Upsilon^c) (\mu_c + \Upsilon_c \sigma) \\ \nabla_a (\rho - \Upsilon^c \mu_c + \Upsilon^c \alpha_c \sigma) - \Upsilon_a (\rho - \Upsilon^c \mu_c + \Upsilon^c \alpha_c \sigma) \\ \quad + (P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b) (\mu^b + \Upsilon^b \sigma) - \hat{T}_a \sigma \end{bmatrix}, \end{aligned}$$

which enjoys some cancellation when expanded, yielding

$$\begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b - J_{ab} \rho + P_{ab} \sigma + \Upsilon_b (\nabla_a \sigma - \mu_a) - J_{ab} \alpha^c \mu_c \\ \nabla_a \rho - (\nabla_a \Upsilon^c) (\mu_c - \alpha_c \sigma) - \Upsilon^c \nabla_a \mu_c + \Upsilon^c \nabla_a (\alpha_c \sigma) - \Upsilon_a \rho \\ - \Upsilon_a \Upsilon^c \alpha_c \sigma + P_{ab} \mu^b - (\nabla_a \Upsilon_b) \mu^b + P_{ab} \Upsilon^b \sigma - (\nabla_a \Upsilon_b) \Upsilon^b \sigma - \hat{T}_a \sigma \end{bmatrix}$$

and, if we substitute for  $\nabla_a \Upsilon^c$  in accordance with (24), then a little more cancellation occurs, yielding

$$\begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b - J_{ab} \rho + P_{ab} \sigma + \Upsilon_b (\nabla_a \sigma - \mu_a) - J_{ab} \alpha^c \mu_c \\ \nabla_a \rho + P_{ab} \mu^b - \hat{T}_a \sigma + 2\Upsilon^b P_{ab} \sigma - \Upsilon^b (\nabla_a \Upsilon_b) \sigma \\ - \alpha^b (\nabla_a \Upsilon_b) \sigma + \Upsilon^b \alpha_b \alpha_a \sigma + \Upsilon^b (\nabla_a \alpha_b) \sigma - \Upsilon_a \Upsilon^b \alpha_b \sigma \\ - \Upsilon^b (\nabla_a \mu_b - J_{ab} \rho + P_{ab} \sigma - J_{ab} \alpha^c \mu_c) + \Upsilon^b \alpha_b (\nabla_a \sigma - \mu_a) \end{bmatrix}.$$

But in Lemma 3 below we show that

$$(25) \quad \begin{aligned} \hat{T}_a &= T_a + 2\Upsilon^b P_{ab} - \Upsilon^b \nabla_a \Upsilon_b \\ &\quad - \alpha^b \nabla_a \Upsilon_b + \Upsilon^b \alpha_b \alpha_a + \Upsilon^b \nabla_a \alpha_b - \Upsilon_a \Upsilon^b \alpha_b \end{aligned}$$

and so this expression reduces to

$$\left[ \begin{array}{c} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b - J_{ab} \rho + P_{ab} \sigma - J_{ab} \alpha^c \mu_c + \Upsilon_b (\nabla_a \sigma - \mu_a) \\ \nabla_a \rho + P_{ab} \mu^b - T_a \sigma \\ - \Upsilon^b (\nabla_a \mu_b - J_{ab} \rho + P_{ab} \sigma - J_{ab} \alpha^c \mu_c) + \Upsilon^b \alpha_b (\nabla_a \sigma - \mu_a) \end{array} \right],$$

which is exactly

$$\widehat{D_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix}} = \widehat{\begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b - J_{ab} \rho + P_{ab} \sigma - J_{ab} \alpha^c \mu_c \\ \nabla_a \rho + P_{ab} \mu^b - T_a \sigma \end{bmatrix}},$$

as required.

Finally, we compute

$$\begin{aligned} &\left\langle D_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix}, \begin{bmatrix} \tilde{\sigma} \\ \tilde{\mu}_c \\ \tilde{\rho} \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix}, D_a \begin{bmatrix} \tilde{\sigma} \\ \tilde{\mu}_c \\ \tilde{\rho} \end{bmatrix} \right\rangle = \\ &\nabla_a (\sigma \tilde{\rho}) - J^{bc} \nabla_a (\mu_b \tilde{\mu}_c) - \alpha^b \mu_b \tilde{\mu}_a + \alpha^c \mu_a \tilde{\mu}_c - \nabla_a (\rho \tilde{\sigma}) = \\ &\nabla_a (\sigma \tilde{\rho}) - J^{bc} \nabla_a (\mu_b \tilde{\mu}_c) - \alpha^b \delta_a^c \mu_b \tilde{\mu}_c + \alpha^c \delta_a^b \mu_b \tilde{\mu}_c - \nabla_a (\rho \tilde{\sigma}) = \\ &\nabla_a (\sigma \tilde{\rho}) - J^{bc} \nabla_a (\mu_b \tilde{\mu}_c) - (\nabla_a J^{bc}) \mu_b \tilde{\mu}_c - \nabla_a (\rho \tilde{\sigma}) = \\ &\nabla_a (\sigma \tilde{\rho} - J^{bc} \mu_b \tilde{\mu}_c - \rho \tilde{\sigma}) = \nabla_a \left\langle \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix}, \begin{bmatrix} \tilde{\sigma} \\ \tilde{\mu}_c \\ \tilde{\rho} \end{bmatrix} \right\rangle, \end{aligned}$$

as required.  $\square$

**Lemma 2.** *The identity (24) holds.*

*Proof.* We compute

$$\nabla_a \Upsilon^b = \nabla_a (J^{bc} \Upsilon_c) = J^{bc} \nabla_a \Upsilon_c + (\nabla_a J^{bc}) \Upsilon_c$$

and we substitute from (9) to conclude that

$$\nabla_a \Upsilon^b = J^{bc} \nabla_a \Upsilon_c + (\alpha^b \delta_a^c - \alpha^c \delta_a^b) \Upsilon_c = J^{bc} \nabla_a \Upsilon_c + \alpha^b \Upsilon_a - \alpha^c \Upsilon_c \delta_a^b,$$

as required.  $\square$

**Lemma 3.** *The identity (25) holds.*

*Proof.* We compute

$$\begin{aligned} 2\hat{\alpha}^b\hat{P}_{ab} &= 2(\alpha^b + \Upsilon^b)(P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b) \\ &= 2\alpha^b P_{ab} + 2\Upsilon^b P_{ab} - 2\Upsilon^b \nabla_a \Upsilon_b - 2\alpha^b \nabla_a \Upsilon_b - 2\Upsilon_a \Upsilon^b \alpha_b \end{aligned}$$

and

$$\begin{aligned} \hat{\alpha}^b \hat{\nabla}_a \hat{\alpha}_b &= (\alpha^b + \Upsilon^b) \nabla_a (\alpha_b + \Upsilon_b) - (\alpha^b + \Upsilon^b) \Upsilon_b (\alpha_a + \Upsilon_a) \\ &= \alpha^b \nabla_a \alpha_b + \Upsilon^b \nabla_a \Upsilon_b \\ &\quad + \alpha^b \nabla_a \Upsilon_b + \Upsilon^b \alpha_b \alpha_a + \Upsilon^b \nabla_a \alpha_b + \Upsilon_a \Upsilon^b \alpha_b. \end{aligned}$$

Adding these two equations gives (25), as required.  $\square$

**Proposition 7.** *The following two homomorphisms  $\mathbb{T} \rightarrow \Lambda^1 \otimes \mathbb{T}$*

$$\begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ \Phi_{ab}\sigma \\ \Phi_{ab}\mu^b + 2(\nabla^b \Phi_{ab})\sigma \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ 0 \\ (\nabla^b \Phi_{ab} + \alpha^a \Phi_{ab})\sigma \end{bmatrix}$$

*are invariantly defined.*

*Proof.* Since  $\hat{\nabla}_c \hat{\Phi}_{ab} = \hat{\nabla}_c \Phi_{ab} = \nabla_c \Phi_{ab} - 2\Upsilon_c \Phi_{ab} - \Upsilon_a \Phi_{cb} - \Upsilon_b \Phi_{ac}$  and  $\hat{\alpha}_a = \alpha_a + \Upsilon_a$  it follows that

$$\hat{\nabla}^b \hat{\Phi}_{ab} = \nabla^b \Phi_{ab} - \Upsilon^b \Phi_{ab} \quad \text{and} \quad \hat{\alpha}^a \hat{\Phi}_{ab} = \alpha^b \Phi_{ab} + \Upsilon^a \Phi_{ab}.$$

The required verifications are immediate.  $\square$

Finally, the *tractor connection* on  $\mathbb{T}$  is defined by modifying  $D_a$  from (23) by appropriate multiples of these homomorphisms. The precise formula is

$$\nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} \equiv \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b - J_{ab} \rho + P_{ab} \sigma - \frac{3}{2n-1} \Phi_{ab} \sigma - J_{ab} \alpha^c \mu_c \\ \nabla_a \rho + P_{ab} \mu^b - \frac{3}{2n-1} \Phi_{ab} \mu^b - \frac{1}{2n+1} (\nabla^b \Phi_{ab}) \sigma \\ \quad - (2\alpha^b P_{ab} + \alpha^b \nabla_a \alpha_b - \frac{10n+7}{(2n+1)(2n-1)} \alpha^b \Phi_{ab}) \sigma \end{bmatrix}.$$

**Theorem 2.** *This connection is well-defined, i.e. is independent of choice of representatives  $(J_{ab}, \nabla_a)$ . It preserves the skew form (22). Its curvature is given by*

$$\begin{aligned} (\nabla_a \nabla_b - \nabla_b \nabla_a) \begin{bmatrix} \sigma \\ \mu_c \\ \rho \end{bmatrix} &= \begin{bmatrix} 0 \\ V_{abcd} \mu^d + Y_{abc} \sigma \\ Y_{abc} \mu^c - \frac{1}{2n} (\nabla^c Y_{abc} - V_{abce} \Phi^{ce}) \sigma \end{bmatrix} \\ &\quad - 2J_{ab} \begin{bmatrix} \rho \\ S_c \sigma - \Phi_{cd} \mu^d \\ S_c \mu^c - \frac{1}{2n} (\Phi_{de} \Phi^{de} + \nabla^c S_c) \sigma \end{bmatrix} \end{aligned}$$

*in Fedosov gauge.*

*Proof.* Mostly, these properties are inherited from the corresponding properties of  $D_a$  as demonstrated in Proposition 6. It only remains to compute its curvature. According to (18) the tractor connection in Fedosov gauge is given by

$$(26) \quad \nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b - J_{ab} \rho + \Phi_{ab} \sigma \\ \nabla_a \rho + \Phi_{ab} \mu^b - S_a \sigma \end{bmatrix},$$

where recall that  $S_a \equiv \frac{1}{2n+1} \nabla^b \Phi_{ab}$ .

We compute

$$\begin{aligned} \nabla_a \nabla_b \begin{bmatrix} \sigma \\ \mu_c \\ \rho \end{bmatrix} &= \nabla_a \begin{bmatrix} \nabla_b \sigma - \mu_b \\ \nabla_b \mu_c - J_{bc} \rho + \Phi_{bc} \sigma \\ \nabla_b \rho + \Phi_{bc} \mu^c - S_b \sigma \end{bmatrix} \\ &= \begin{bmatrix} \nabla_a (\nabla_b \sigma - \mu_b) - (\nabla_b \mu_a - J_{ba} \rho + \Phi_{ba} \sigma) \\ \nabla_a (\nabla_b \mu_c - J_{bc} \rho + \Phi_{bc} \sigma) - J_{ac} (\nabla_b \rho + \Phi_{bd} \mu^d - S_b \sigma) \\ \nabla_a (\nabla_b \rho + \Phi_{bc} \mu^c - S_b \sigma) - \Phi_a^c (\nabla_b \mu_c - J_{bc} \rho + \Phi_{bc} \sigma) - S_a (\nabla_b \sigma - \mu_b) \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} (\nabla_a \nabla_b - \nabla_b \nabla_a) \begin{bmatrix} \sigma \\ \mu_c \\ \rho \end{bmatrix} &= \begin{bmatrix} -2J_{ab} \rho \\ (\nabla_a \nabla_b - \nabla_b \nabla_a) \mu_c - J_{ac} \Phi_{bd} \mu^d - \Phi_{ac} \mu_b + J_{bc} \Phi_{ad} \mu^d + \Phi_{bc} \mu_a \\ (\nabla_a \Phi_{bc} - \nabla_b \Phi_{ac} + J_{ac} S_b - J_{bc} S_a) \mu^c - (\nabla_a S_b - \nabla_b S_a + 2\Phi_a^c \Phi_{bc}) \sigma \end{bmatrix}. \end{aligned}$$

However, from (15) we see that

$$\begin{aligned} (\nabla_a \nabla_b - \nabla_b \nabla_a) \mu_c &= R_{abcd} \mu^d \\ &= V_{abcd} \mu^d + J_{ac} \Phi_{bd} \mu^d - J_{bc} \Phi_{ad} \mu^d - \Phi_{bc} \mu_a + \Phi_{ac} \mu_b + 2J_{ab} \Phi_{cd} \mu^d \end{aligned}$$

and, if we also substitute from (20), then we obtain

$$\begin{aligned} (\nabla_a \nabla_b - \nabla_b \nabla_a) \begin{bmatrix} \sigma \\ \mu_c \\ \rho \end{bmatrix} &= \begin{bmatrix} -2J_{ab} \rho \\ V_{abcd} \mu^d + 2J_{ab} \Phi_{cd} \mu^d + (Y_{abc} - 2J_{ab} S_c) \sigma \\ (Y_{abc} - 2J_{ab} S_c) \mu^c - (\nabla_a S_b - \nabla_b S_a + 2\Phi_a^c \Phi_{bc}) \sigma \end{bmatrix}. \end{aligned}$$

Lemma 4 below allows us to rewrite this expression as

$$\left[ \begin{array}{c} -2J_{ab}\rho \\ V_{abcd}\mu^d + 2J_{ab}\Phi_{cd}\mu^d + (Y_{abc} - 2J_{ab}S_c)\sigma \\ (Y_{abc} - 2J_{ab}S_c)\mu^c - \frac{1}{2n}(\nabla^c Y_{abc} - V_{abce}\Phi^{ce} - 2J_{ab}(\Phi_{de}\Phi^{de} + \nabla^c S_c))\sigma \end{array} \right],$$

as required.  $\square$

**Lemma 4.** *The identity*

$$\nabla^c Y_{abc} = V_{abce}\Phi^{ce} + 2J_{ab}(\Phi_{de}\Phi^{de} + \nabla^c S_c) + 2n(\nabla_a S_b - \nabla_b S_a + 2\Phi_a{}^c\Phi_{bc})$$

*holds in Fedosov gauge.*

*Proof.* Using (10) to commute derivatives

$$\begin{aligned} \nabla_d \nabla_a \Phi_{bc} &= (\nabla_d \nabla_a - \nabla_a \nabla_d)\Phi_{bc} + \nabla_a \nabla_d \Phi_{bc} \\ &= R_{dabe}\Phi^e{}_c + R_{dace}\Phi_b{}^e + \nabla_a \nabla_d \Phi_{bc}, \end{aligned}$$

whence

$$\begin{aligned} \nabla^c \nabla_a \Phi_{bc} &= R^c{}_{abe}\Phi^e{}_c + R^c{}_{ace}\Phi_b{}^e + \nabla_a \nabla^c \Phi_{bc} \\ &= R^c{}_{abe}\Phi^e{}_c + R^c{}_{ace}\Phi_b{}^e + (2n+1)\nabla_a S_b. \end{aligned}$$

Substituting from (15) gives

$$\nabla^c \nabla_a \Phi_{bc} = V_{acbe}\Phi^{ce} + 2n\Phi_a{}^c\Phi_{bc} + J_{ab}\Phi_{de}\Phi^{de} + (2n+1)\nabla_a S_b$$

and, similarly,

$$\nabla^c \nabla_b \Phi_{ac} = V_{bcae}\Phi^{ce} - 2n\Phi_a{}^c\Phi_{bc} - J_{ab}\Phi_{de}\Phi^{de} + (2n+1)\nabla_b S_a.$$

Noting that  $V_{acbe} - V_{bcae} = V_{abce}$ , we may subtract these two equations to obtain

$$\begin{aligned} \nabla^c \nabla_a \Phi_{bc} - \nabla^c \nabla_b \Phi_{ac} \\ = V_{abce}\Phi^{ce} + 4n\Phi_a{}^c\Phi_{bc} + 2J_{ab}\Phi_{de}\Phi^{de} + (2n+1)(\nabla_a S_b - \nabla_b S_a). \end{aligned}$$

Therefore, from the formula (20) for  $Y_{abc}$ , we conclude that

$$\begin{aligned} \nabla^c Y_{abc} &= \nabla^c \nabla_a \Phi_{bc} - \nabla^c \nabla_b \Phi_{ac} - (\nabla_a S_b - \nabla_b S_a) + 2J_{ab}\nabla^c S_c \\ &= V_{abce}\Phi^{ce} + 4n\Phi_a{}^c\Phi_{bc} + 2J_{ab}\Phi_{de}\Phi^{de} \\ &\quad + 2n(\nabla_a S_b - \nabla_b S_a) + 2J_{ab}\nabla^c S_c, \end{aligned}$$

as required.  $\square$

Theorem 2 has the following immediate consequence.

**Corollary 2.** *The curvature of the tractor connection has the form*

$$(27) \quad (\nabla_a \nabla_b - \nabla_b \nabla_a)\Sigma = 2J_{ab}\Theta\Sigma$$

*for some endomorphism  $\Theta$  of  $\mathbb{T}$  if and only if  $V_{abcd} \equiv 0$ .*

*Proof.* Notice that the curvature in the statement of Theorem 2 is split already into its irreducible components according to

$$\Lambda^2 \otimes \text{End}(\mathbb{T}) = \Lambda_{\perp}^2 \otimes \text{End}(\mathbb{T}) \oplus \text{End}(\mathbb{T}),$$

where  $\Lambda_{\perp}^2$  denotes the 2-forms that are trace-free with respect to  $J_{ab}$ . That the curvature has the form (27) is precisely that the component in  $\Lambda_{\perp}^2 \otimes \text{End}(\mathbb{T})$  vanish, i.e. that

$$\begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ V_{abcd}\mu^d + Y_{abc}\sigma \\ Y_{abc}\mu^c - \frac{1}{2n}(\nabla^c Y_{abc} - V_{abce}\Phi^{ce})\sigma \end{bmatrix}$$

vanish identically. Clearly this implies that  $V_{abcd} \equiv 0$  but then the contracted Bianchi identity (19) implies that  $Y_{abc} \equiv 0$ .  $\square$

We may further pursue the consequences of  $V_{abcd} = 0$  as follows.

**Lemma 5.** *When (27) holds and the homomorphism  $\Theta$  is written in Fedosov gauge, then  $\nabla_a \Theta = 0$ .*

*Proof.* When (27) holds, the Bianchi identity for the connection  $\nabla_a$  on  $\mathbb{T}$  implies that  $\nabla_{[a}(J_{bc]}\Theta) = 0$ . In Fedosov gauge, this may be rewritten as  $J_{[bc}\nabla_{a]}\Theta = 0$ . Non-degeneracy of  $J_{bc}$  implies that  $\nabla_a \Theta = 0$ .  $\square$

From Theorem 2, when (27) holds the homomorphism  $\Theta$  is given in Fedosov gauge by

$$\Theta \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} -\rho \\ \Phi_{bc}\mu^c - S_b\sigma \\ X\sigma - S_c\mu^c \end{bmatrix}, \quad \text{where } X \equiv \frac{1}{2n}(\Phi_{de}\Phi^{de} + \nabla^c S_c).$$

But, by using the invariant symplectic form (22) on  $\mathbb{T}$ , we can equally well view  $\Theta$  as a section of  $\otimes^2 \mathbb{T}$ . Specifically,

$$(28) \quad \Theta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\Phi_{bc} & S_b \\ 0 & S_c & -X \end{bmatrix}.$$

Note that  $\Theta$  is symmetric (as must be the case since  $\nabla_a$  preserves the symplectic form (22) on  $\mathbb{T}$ ).

**Theorem 3.** *If  $V_{abcd} = 0$ , then*

$$(29) \quad (\nabla_a \Phi^{bc})_{\circ} = 0$$

*in Fedosov gauge, where  $(\ )_{\circ}$  means to take the trace-free part.*



*Proof.* From (26) and (28) we compute

$$\nabla_a \Theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\nabla_a \Phi_{bc} - J_{ab} S_c - J_{ac} S_b & \nabla_a S_b + J_{ab} X - \Phi_{ac} \Phi_b^c \\ 0 & \nabla_a S_c - \Phi_{ab} \Phi_c^b + J_{ac} X & -\nabla_a X + \Phi_{ab} S^b + \Phi_{ac} S^c \end{bmatrix}.$$

From Lemma 5 we conclude that

$$(30) \quad \nabla_a \Phi_{bc} + J_{ab} S_c + J_{ac} S_b = 0$$

and raising indices with  $J^{ab}$  gives

$$\nabla_a \Phi^{bc} + \delta_a^b S^c + \delta_a^c S^b = 0,$$

as required.  $\square$

Several remarks are in order. Firstly, notice that (30) is only an extra condition on  $\nabla_{[a} \Phi_{b]c}$  since  $\nabla_{[a} \Phi_{b]c} + J_{ab} S_c - J_{c[a} S_{b]} = \frac{1}{2} Y_{abc}$  in accordance with (20). Secondly, the partial differential equations (29) are the well-known *mobility* equations [10] of projective differential geometry whose non-degenerate solutions  $\Phi^{ab}$  are in one-to-one correspondence with (pseudo-)metrics having connection in the projective class  $[\nabla_a]$  of  $\nabla_a$ . Thirdly, the other components of  $\nabla_a \Theta$  apparently give rise to a whole system of equations,

$$\begin{aligned} \nabla_a \Phi^{bc} + \delta_a^b S^c + \delta_a^c S^b &= 0 \\ \nabla_a S^b + \delta_a^b X - \Phi_{ac} \Phi^{bc} &= 0 \\ \nabla_a X - 2\Phi_{ab} S^b &= 0 \end{aligned}$$

but, in fact, this is exactly the prolongation of the (29) as derived in [7]. Therefore, the vanishing of  $\nabla_a \Theta$  is precisely equivalent to the mobility equations (29) on  $\Phi^{ab}$ . As described in [7], for (29) to admit any non-zero solutions imposes further non-trivial conditions on the projective structure  $[\nabla_a]$ . If  $\Phi_{ab} \equiv 0$ , however, then the connection  $\nabla_a$  is flat, as can be seen from (15). Finally, notice that the partial differential equations (29) are actually much stronger than the mobility equations alone because  $\Phi^{ab}$  is actually part of the curvature of  $\nabla_a$ .

**7.1. Examples.** In view of the strength of equations (29) it is not easy to provide any non-trivial examples of a conformally Fedosov structure with  $V_{abcd} = 0$ . Complex projective space  $\mathbb{CP}_n$  with its usual projective structure and symplectic form certainly provides the best example. In this case, recall that

$$V_{abcd} = 0 \quad \text{and} \quad \Phi_{ab} = g_{ab}$$

so  $\mathbb{CP}_n$  is not projectively flat. In Fedosov gauge  $S_a = 0$  and

$$\Theta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -g_{bc} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

as a section of  $\odot^2\mathbb{T}$ .

Another example may be based on  $S^1 \times S^{2n-1}$  with its conformally symplectic structure induced by the dilation invariant

$$J_{ab} \equiv (1/\|x\|)^2 \omega_{ab}$$

on  $\mathbb{R}^{2n} \setminus \{0\}$ , where  $\omega_{ab}$  is the standard symplectic form

$$\omega_{ab} dx^a dx^b = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \cdots.$$

The flat connection  $\partial_a$  on  $\mathbb{R}^{2n} \setminus \{0\}$  is also dilation invariant whence the pair  $(J_{ab}, \partial_a)$  defines a dilation invariant conformally Fedosov structure on  $\mathbb{R}^{2n} \setminus \{0\}$ . Indeed,

$$\partial_a J_{bc} = 2(\partial_a \log(1/\|x\|)) J_{bc} = -2(x_a/\|x\|^2) J_{bc}$$

so (5) holds with  $\alpha_a = -x_a/\|x\|^2$  and  $\beta_a = -2x_a/\|x\|^2$ . The proof of Proposition 3 shows that we should take

$$\nabla_a \phi_b = \partial_a \phi_b + x_a \phi_b / \|x\|^2 + x_b \phi_a / \|x\|^2$$

as the unique projectively flat connection so that (7) holds. Notice that, although

$$V_{abcd} = 0 \quad \text{and} \quad \Phi_{ab} = 0$$

in this case, the curvature of the corresponding tractor connection is not flat. Indeed, we have  $(\nabla_a \nabla_b - \nabla_b \nabla_a) \Sigma = 2J_{ab} \Theta \Sigma$  where, as a section of  $\odot^2\mathbb{T}$ ,

$$\Theta = \begin{bmatrix} -1 & x_c/\|x\|^2 & -1/\|x\|^2 \\ x_b/\|x\|^2 & -x_b x_c / \|x\|^4 & x_b / \|x\|^4 \\ -1/\|x\|^2 & x_c / \|x\|^4 & -1/\|x\|^4 \end{bmatrix},$$

which has rank 1.

## REFERENCES

- [1] A. Banyaga, *Symplectic geometry and related structures*, Cubo **6** (2004) 123–138.
- [2] T.P. Branson, A. Čap, M.G. Eastwood, and A.R. Gover, *Prolongations of geometric overdetermined systems*, Internat. Jour. Math. **17** (2006) 641–664.
- [3] T.N. Bailey, M.G. Eastwood, and A.R. Gover, *Thomas's structure bundle for conformal, projective and related structures*, Rocky Mountain Jour. Math. **24** (1994) 1191–1217.

- [4] É. Cartan, *Les espaces à connexion conforme*, Ann. Soc. Pol. Math. **2** (1923) 171–202.
- [5] A. Čap and J. Slovák, *Parabolic Geometries I: Background and General Theory*, Math. Surv. and Monographs **154**, Amer. Math. Soc. 2009.
- [6] M.G. Eastwood, *Notes on projective differential geometry*, Symmetries and Overdetermined Systems of Partial Differential Equations, IMA Volumes No. 144, Springer 2008, pp. 41–60.
- [7] M.G. Eastwood and V.S. Matveev, *Metric connections in projective differential geometry*, Symmetries and Overdetermined Systems of Partial Differential Equations, IMA Volumes No. 144, Springer 2008, pp. 339–350.
- [8] I.M. Gelfand, V.S. Retakh, and M.A. Shubin, *Fedosov manifolds*, Adv. Math. **136** (1998) 104–140.
- [9] H.C. Lee, *A kind of even-dimensional differential geometry and its application to exterior calculus*, Amer. Jour. Math. **65** (1943) 433–438.
- [10] J. Mikeš, *Geodesic mappings of affine-connected and Riemannian spaces*, Jour. Math. Sci. **78** (1996) 311–333.
- [11] R. Penrose and W. Rindler, *Spinors and Space-time, vol. 1*, Cambridge University Press 1984.
- [12] T.Y. Thomas, *On conformal geometry*, Proc. Nat. Acad. Sci. **12** (1926) 352–359.
- [13] I. Vaisman, *Locally conformal symplectic manifolds*, Internat. Jour. Math. Math. Sci. **8** (1985) 521–536.
- [14] W.J. Westlake, *Conformally Kähler manifolds*, Math. Proc. Camb. Phil. Soc. **50** (1954) 16–19.

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